Digital Communication Systems
ECS 452

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3 Discrete Memoryless Channel (DMC)

Office Hours:
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Monday	10:00-10:40
Tuesday	12:00-12:40
Thursday	14:20-15:30

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3.1 DMC Models

Summary: DMC

Channel Input Alphabet $S_X \equiv \mathcal{X}$

$S_Y \equiv \mathcal{Y}$

$$P[X = x] \equiv p_X(x) \equiv p(x) \Rightarrow \underline{\mathbf{p}}$$ row vector

$$P[Y = y] \equiv p_Y(y) \equiv q(y) \Rightarrow \underline{\mathbf{q}}$$ row vector

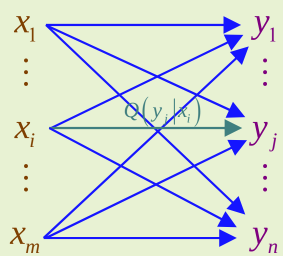
$$P[Y = y|X = x] \equiv p_{Y|X}(y|x) \equiv Q(y|x) \Rightarrow \mathbf{Q} \text{ matrix}$$

Put together All values at corresponding positions in the matrix.

$$P[X = x, Y = y] \equiv P_{X,Y}(x, y) \equiv P(x, y) \Rightarrow \mathbf{P} \text{ matrix}$$

"AND"

$$\mathbf{Q} = \begin{array}{c} {}_{x}\diagdown^{y} \quad \mathcal{Y} \\ \begin{matrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{matrix} \begin{bmatrix} \ddots & & \cdot^{\cdot^{\cdot}} \\ & Q(y_j|x_i) & \\ \cdot^{\cdot^{\cdot}} & & \ddots \end{bmatrix} \end{array} \begin{matrix} \xrightarrow{\times p(x_1)} \\ \vdots \\ \xrightarrow{\times p(x_i)} \\ \vdots \\ \xrightarrow{\times p(x_m)} \end{matrix} \begin{array}{c} {}_{x}\diagdown^{y} \; y_1 \cdots y_j \cdots y_n \\ \begin{matrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{matrix} \begin{bmatrix} \ddots & & \cdot^{\cdot^{\cdot}} \\ & p_{X,Y}(x_i, y_j) & \\ \cdot^{\cdot^{\cdot}} & & \ddots \end{bmatrix} \end{array} = \mathbf{P}$$



$$\Sigma \downarrow \quad \Sigma \downarrow \quad \Sigma \downarrow$$

$$\begin{bmatrix} & q(y_j) & \end{bmatrix} = \underline{\mathbf{q}} = \underline{\mathbf{p}}\mathbf{Q}$$

Review: Evaluation of Probability from the Joint PMF Matrix

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X+Y<7]$

Step 1: Find the pairs (x,y) that satisfy the condition "$x+y<7$"

One way to do this is to first construct the matrix of $x+y$.

$x+y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

Review: Evaluation of Probability from the Joint PMF Matrix

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X+Y<7]$

Step 2: Add the corresponding probabilities from the joint pmf (matrix)

$$P[X+Y<7] = 0.1+0.1+0.1$$
$$= 0.3$$

$x+y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

Review: Evaluation of Probability from the Joint PMF Matrix

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X = Y]$

$$P[X = Y] = 0 + 0.2 + 0.3 = 0.5$$

Review: Sum of two discrete RVs

- Consider two random variables X and Y.
- Suppose their **joint pmf matrix** is

$P_{X,Y}$

x \ y	2	3	4	5	6
1	0.1	0.1	0	0	0
3	0.1	0	0	0.1	0
4	0	0.1	0.2	0	0
6	0	0	0	0	0.3

- Find $P[X+Y=7]$

$P[X+Y=7] = 0.1$

$x+y$

x \ y	2	3	4	5	6
1	3	4	5	6	7
3	5	6	7	8	9
4	6	7	8	9	10
6	8	9	10	11	12

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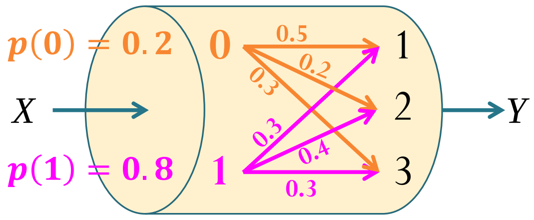
3.2 Decoder and $P(\mathcal{E})$

Recipe for finding $P(\mathcal{E})$ of any decoder



y	$\hat{x}(y)$
1	1
2	1
3	0

$$\mathbf{Q} = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 0 & 0.5 & 0.2 & 0.3 \\ 1 & 0.3 & 0.4 & 0.3 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.8} \end{array} \quad \begin{array}{ccc|c} 1 & 1 & 0 & \hat{x}(y) \\ 1 & 2 & 3 & y / x \\ \hline 0.10 & 0.04 & \boxed{0.06} & 0 \\ \boxed{0.24} & \boxed{0.32} & 0.24 & 1 \end{array} = \mathbf{P}$$

- Use the **P** matrix.
  - If unavailable, can be found by scaling each row of the **Q** matrix by its corresponding $p(x)$.
- Write $\hat{x}(y)$ values on top of the y values for the **P** matrix.
- For column y in the **P** matrix, circle the element whose corresponding x value is the same as $\hat{x}(y)$.
- $P(\mathcal{C}) =$ the sum of the circled probabilities.
- $P(\mathcal{E}) = 1 - P(\mathcal{C})$.

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3.3 Optimal Decoder

Review: ECS315 (2017)

6.4. ***Interpretation***: It is sometimes useful to interpret $P(A)$ as our knowledge of the occurrence of event A *before* the experiment takes place. Conditional probability[25] $P(A|B)$ is the ***updated probability*** of the event A given that we now know that B occurred (but we still do not know which particular outcome in the set B did occur).

Definition 6.5. Sometimes, we refer to $P(A)$ as	in which case, we refer to $P(A|B)$ as
• **a priori probability**, or	a posteriori probability
• the **prior probability** of A, or	the posterior (probability)
• the **unconditional probability** of A.	conditional probability

Guessing Game 1

- There are 15 cards.
  - Each have a number on it.
  - Here are the 15 cards:

1 2 2 3 3 3 4 4 4 4 5 5 5 5 5

- One card is randomly selected from the 15 cards.
- You need to guess the number on the card.
- Have to pay 1 Baht for incorrect guess.

- The game is to be repeated $n = 10{,}000$ times.
- What should be your guess value?

```
close all; clear all;

n = 5; % number of time to play this game

D = [1 2 2 3 3 3 4 4 4 4 5 5 5 5 5];
X = D(randi(length(D),1,n));

if n <= 10
    X
end

g = 1
cost = sum(X ~= g)

if n > 1
averageCostPerGame = cost/n
end
```

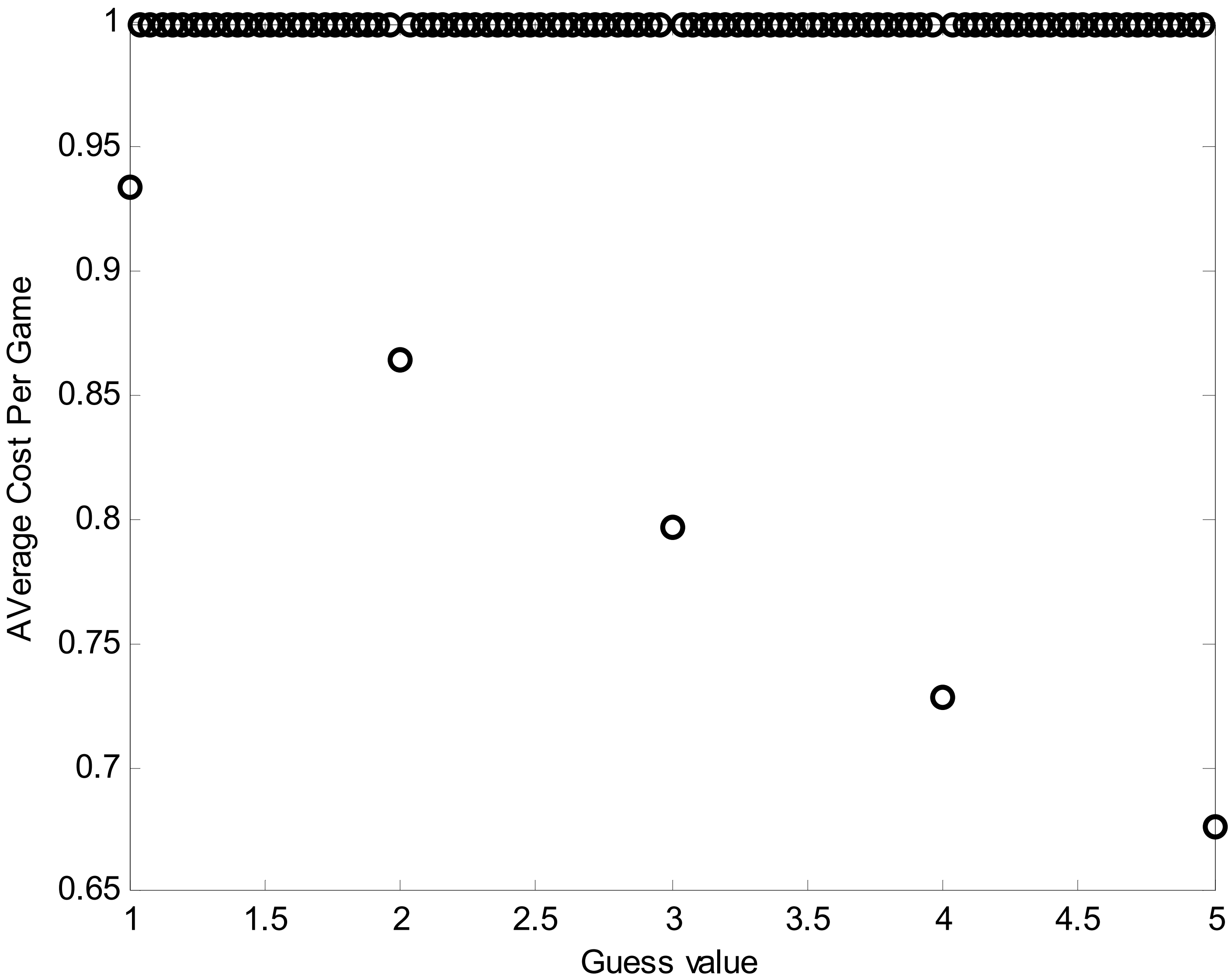
```
>> GuessingGame_4_1_1
X =
     3     5     1     2     5
g =
     1
cost =
     4
averageCostPerGame =
    0.8000
```

```
close all; clear all;

n = 5; % number of time to play this game

D = [1 2 2 3 3 3 4 4 4 4 5 5 5 5 5];
X = D(randi(length(D),1,n));

if n <= 10
    X
end

g = 3.3
cost = sum(X ~= g)

if n > 1
averageCostPerGame = cost/n
end
```

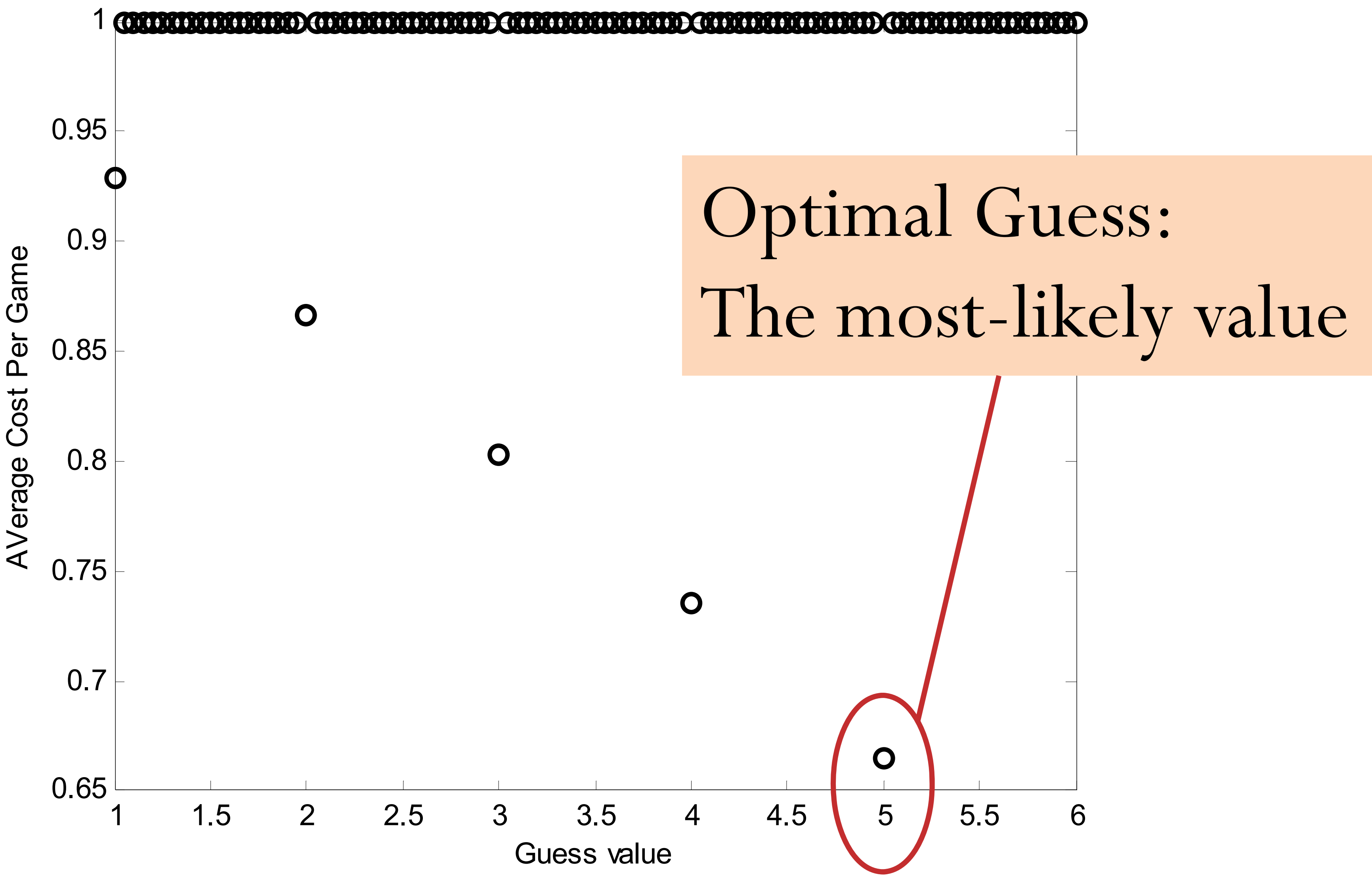
```
>> GuessingGame_4_1_1
X =
     5     3     2     4     1
g =
    3.3000
cost =
     5
averageCostPerGame =
     1
```

```
close all; clear all;

n = 1e4; % number of time to play this game

D = [1 2 2 3 3 3 4 4 4 4 5 5 5 5 5];
X = D(randi(length(D),1,n));

if n <= 10
    X
end

g = ?
cost = sum(X ~= g)

if n > 1
averageCostPerGame = cost/n
end
```

Guessing Game 1



Guessing Game 1



Summary: MAP vs. ML Decoders

$$\hat{x}_{\text{MAP}}(y) = \arg\max_x p_{X,Y}(x,y) = \hat{x}_{\text{optimal}}(y) = \arg\max_x \underbrace{P[X = x | Y = y]}_{\text{a posteriori probability}} = \arg\max_x Q(y|x) \underbrace{p(x)}_{\text{prior probability}}$$

- Decoder is derived from the **P** matrix
- Select (by circling) the maximum value in each column (for each value of y) in the **P** matrix.
- The corresponding x value is the value of $\hat{x}_{\text{MAP}}(y)$.

$$\hat{x}_{\text{ML}}(y) = \arg\max_x \underbrace{Q(y|x)}_{\text{likelihood funtion}}$$

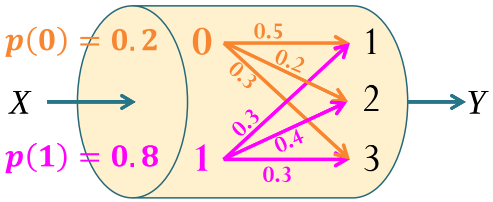
Optimal at least when $p(x)$ is uniform (the channel inputs are equally likely)

Can be derived without knowing the channel input probabilities.

- Decoder is derived from the **Q** matrix
- Select (by circling) the maximum value in each column (for each value of y) in the **Q** matrix.
- The corresponding x value is the value of $\hat{x}_{\text{ML}}(y)$.

- Once the decoder (the decoding table) is derived $P(\mathcal{C})$ and $P(\mathcal{E})$ are calculated from adding the corresponding probabilities in the **P** matrix.

Example: MAP Decoder [Ex. 3.36]



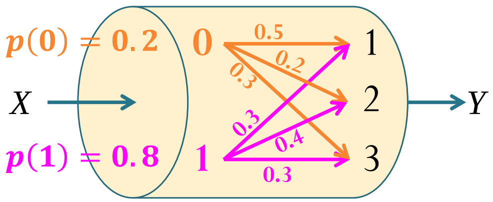
y	$\hat{x}_{\text{MAP}}(y)$
1	1
2	1
3	1

$$\hat{x}_{\text{MAP}}(y): \quad 1 \quad 1 \quad 1$$

$$\mathbf{Q} = \begin{array}{c} x \backslash y \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.8} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.10 & 0.04 & 0.06 \\ \boxed{0.24} & \boxed{0.32} & \boxed{0.24} \end{bmatrix} \end{array} \begin{array}{c} y / x \\ 0 \\ 1 \end{array} = \mathbf{P}$$

$$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - (0.24 + 0.32 + 0.24) = 0.2$$

Example: ML Decoder [Ex. 3.47]



[Same as Ex. 3.27]

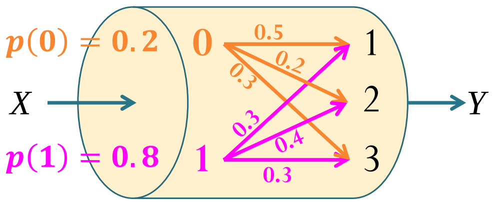
y	$\hat{x}_{ML}(y)$
1	0
2	1
3	0

Sol 1:

$$\mathbf{Q} = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ 0 & (0.5) & 0.2 & (0.3) \\ 1 & 0.3 & (0.4) & 0.3 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.8} \end{array} \quad \begin{array}{ccc|c} 1 & 2 & 3 & y/x \\ (0.10) & 0.04 & (0.06) & 0 \\ 0.24 & (0.32) & 0.24 & 1 \end{array} = \mathbf{P}$$

$\hat{x}_{ML}(y)$: 0, 1, 0

$$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - (0.10 + 0.32 + 0.06) = 0.52$$



y	$\hat{x}_{ML}(y)$
1	0
2	1
3	1

Sol 2:

$$\mathbf{Q} = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ 0 & (0.5) & 0.2 & 0.3 \\ 1 & 0.3 & (0.4) & (0.3) \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.8} \end{array} \quad \begin{array}{ccc|c} 1 & 2 & 3 & y/x \\ (0.10) & 0.04 & 0.06 & 0 \\ 0.24 & (0.32) & (0.24) & 1 \end{array} = \mathbf{P}$$

$\hat{x}_{ML}(y)$: 0, 1, 1

[Agree with Ex. 3.33]

$$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - (0.10 + 0.32 + 0.24) = 0.34$$

MAP Decoder

```
%% MAP Decoder
P = diag(p_X)*Q; % Weight the channel transition probability by the
                 % corresponding prior probability.
[V I] = max(P); % For I, the default MATLAB behavior is that when there are
                % multiple max, the index of the first one is returned.
Decoder_Table = S_X(I) % The decoded values corresponding to the received Y
```

```
%% Decode according to the decoder table
x_hat = y; % preallocation
for k = 1:length(S_Y)
    I = (y==S_Y(k));
    x_hat(I) = Decoder_Table(k);
end

PE_sim = 1-sum(x==x_hat)/n % Error probability from the simulation
```

```
%% Calculation of the theoretical error probability
PC = 0;
for k = 1:length(S_X)
    I = (Decoder_Table == S_X(k));
    Q_row = Q(k,:);
    PC = PC+ p_X(k)*sum(Q_row(I));
end
PE_theretical = 1-PC
```


[DMC_decoder_MAP_demo.m]

ML Decoder

```
%% ML Decoder
[V I] = max(Q); % For I, the default MATLAB behavior is that when there are
                % multiple max, the index of the first one is returned.
Decoder_Table = S_X(I) % The decoded values corresponding to the received Y
```

```
%% Decode according to the decoder table
x_hat = y; % preallocation
for k = 1:length(S_Y)
    I = (y==S_Y(k));
    x_hat(I) = Decoder_Table(k);
end

PE_sim = 1-sum(x==x_hat)/n % Error probability from the simulation
```

```
%% Calculation of the theoretical error probability
PC = 0;
for k = 1:length(S_X)
    I = (Decoder_Table == S_X(k));
    Q_row = Q(k,:);
    PC = PC+ p_X(k)*sum(Q_row(I));
end
PE_theretical = 1-PC
```


[DMC_decoder_ML_demo.m]

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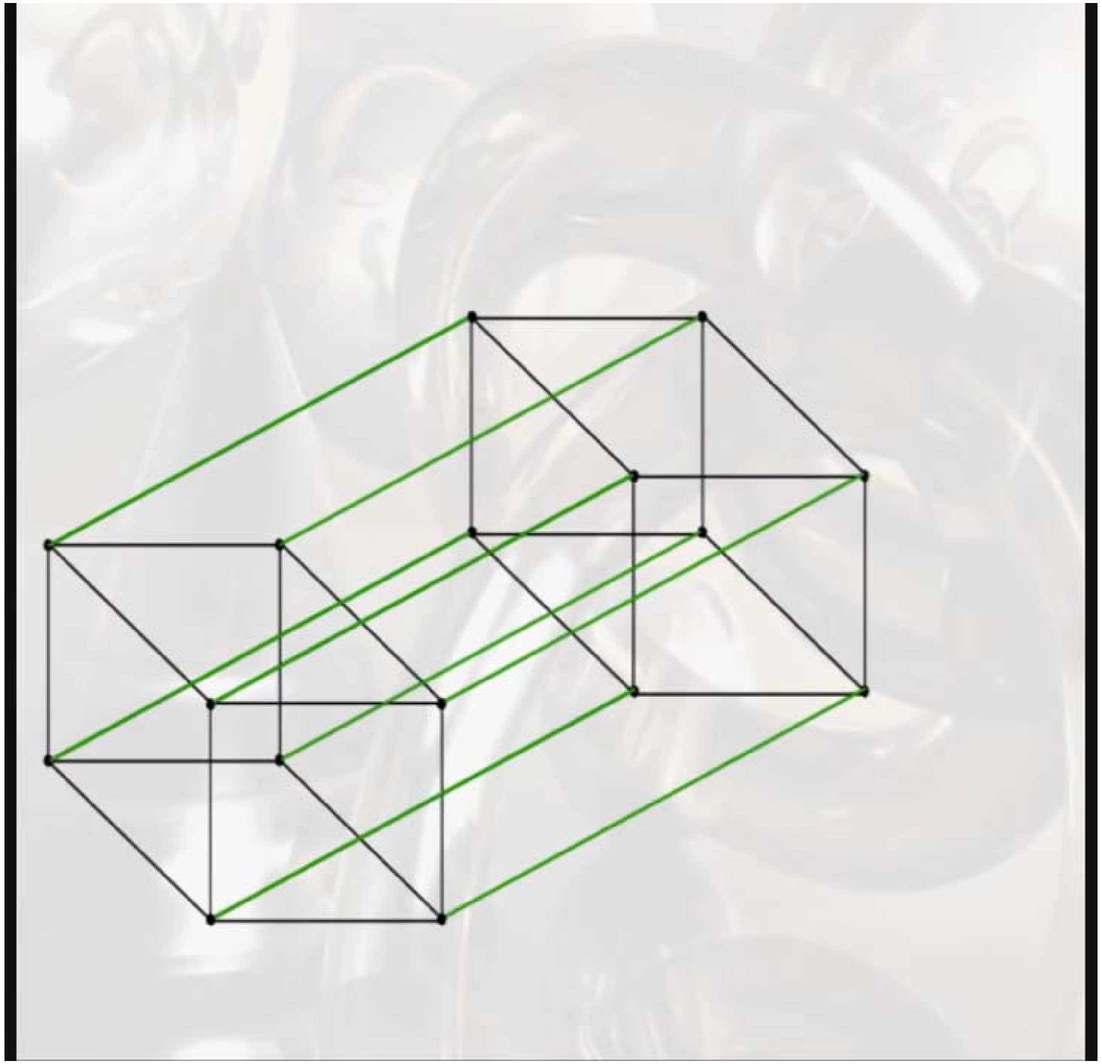
3.5 Repetition Code in Communications Over BSC

Hypercube

https://www.youtube.com/watch?v=Q_B5GpsbSQw

Hypercube



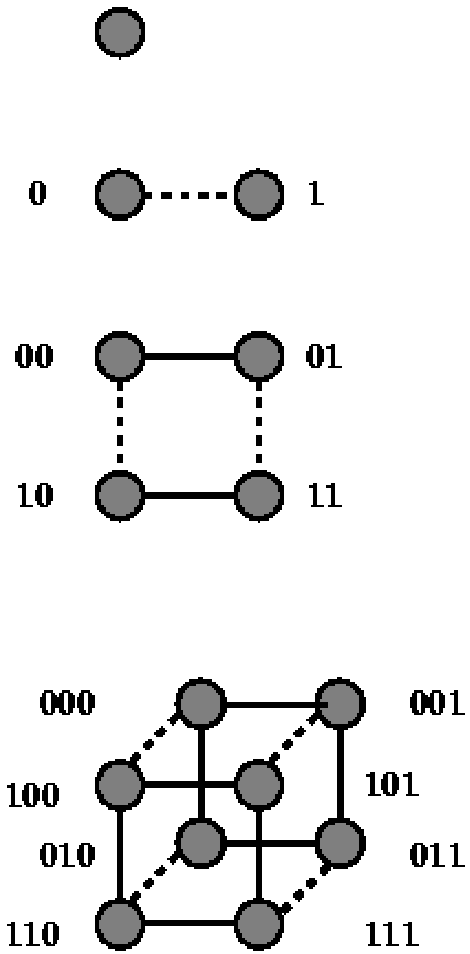
https://www.youtube.com/watch?v=Q_B5GpsbSQw

n-bit space

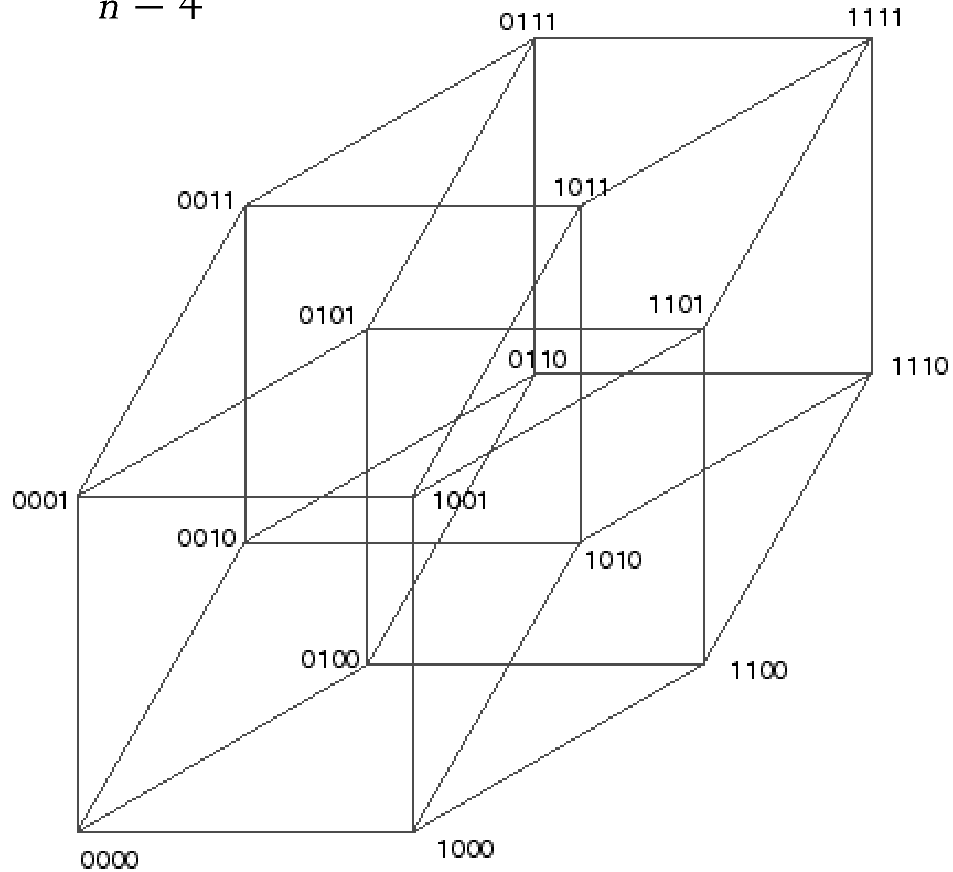
$n = 1$

$n = 2$

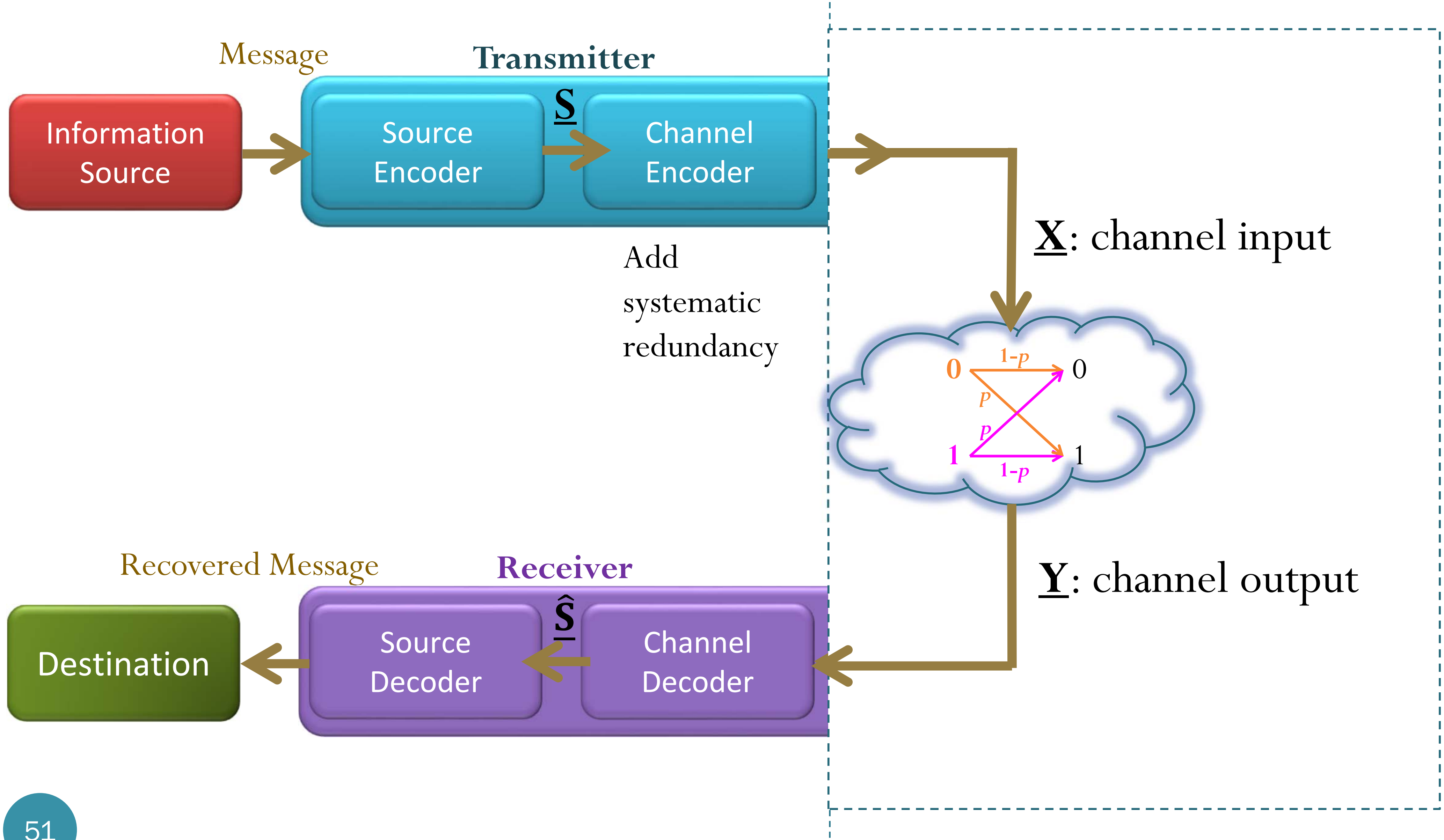
$n = 3$



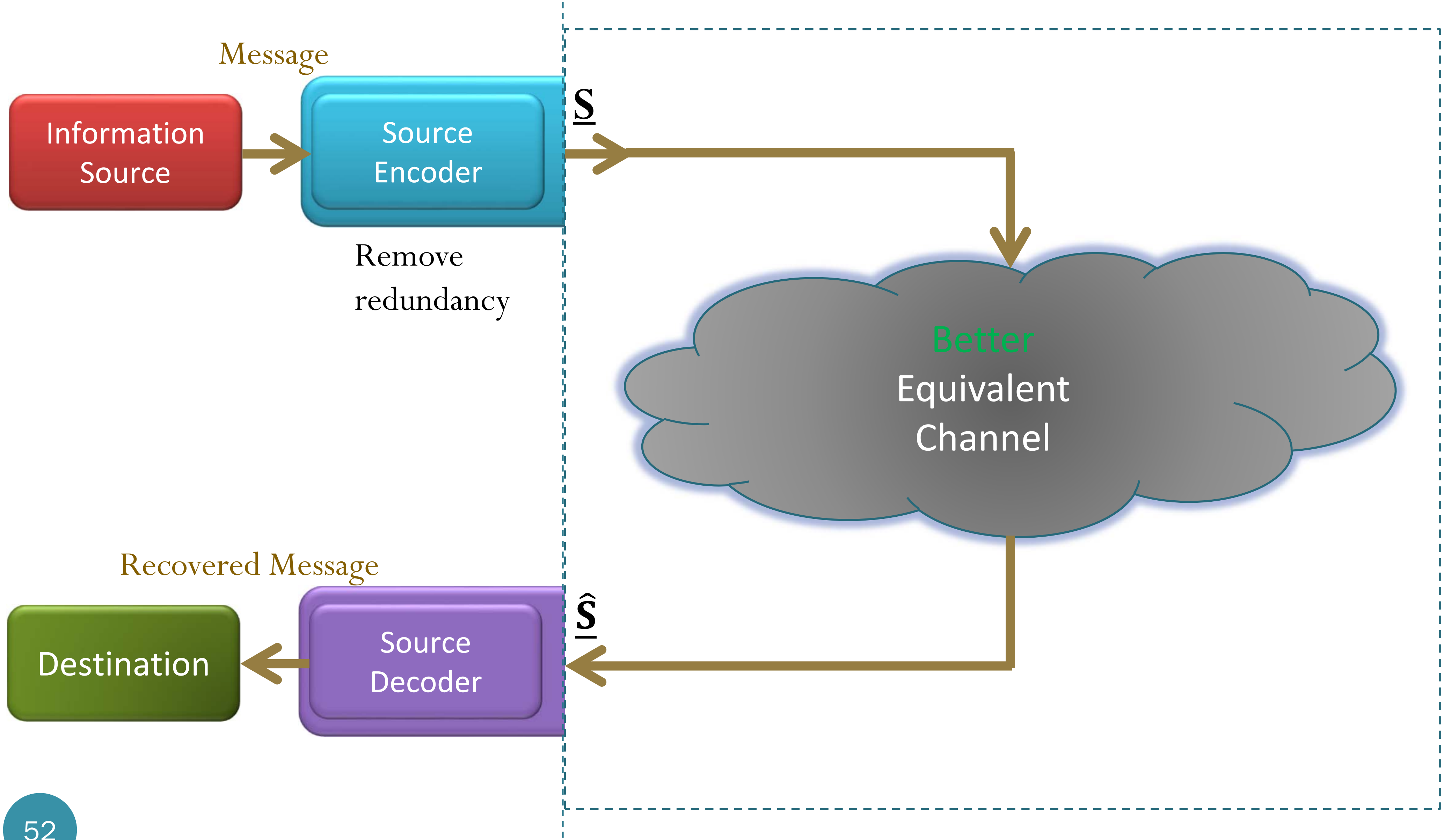
$n = 4$



Channel Encoder and Decoder

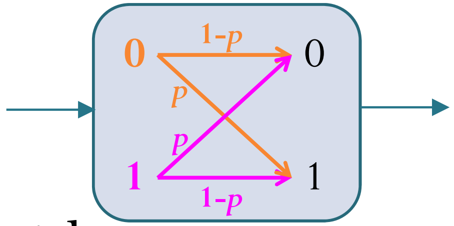


Better Equivalent Channel

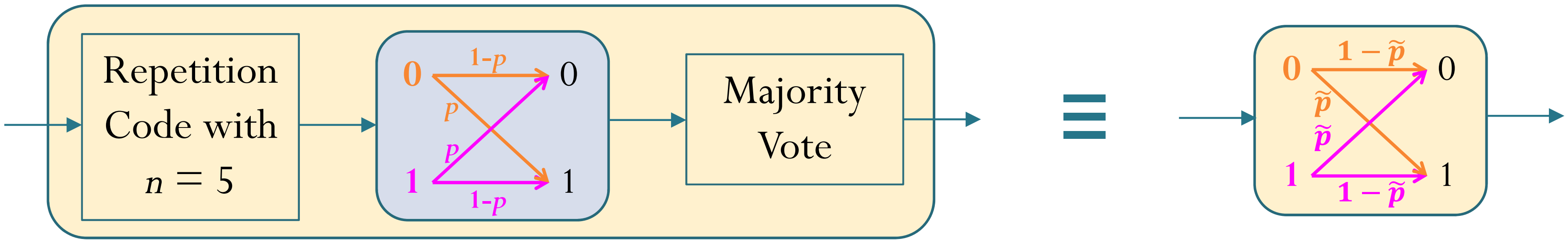


Example: Repetition Code

- Original Equivalent Channel:

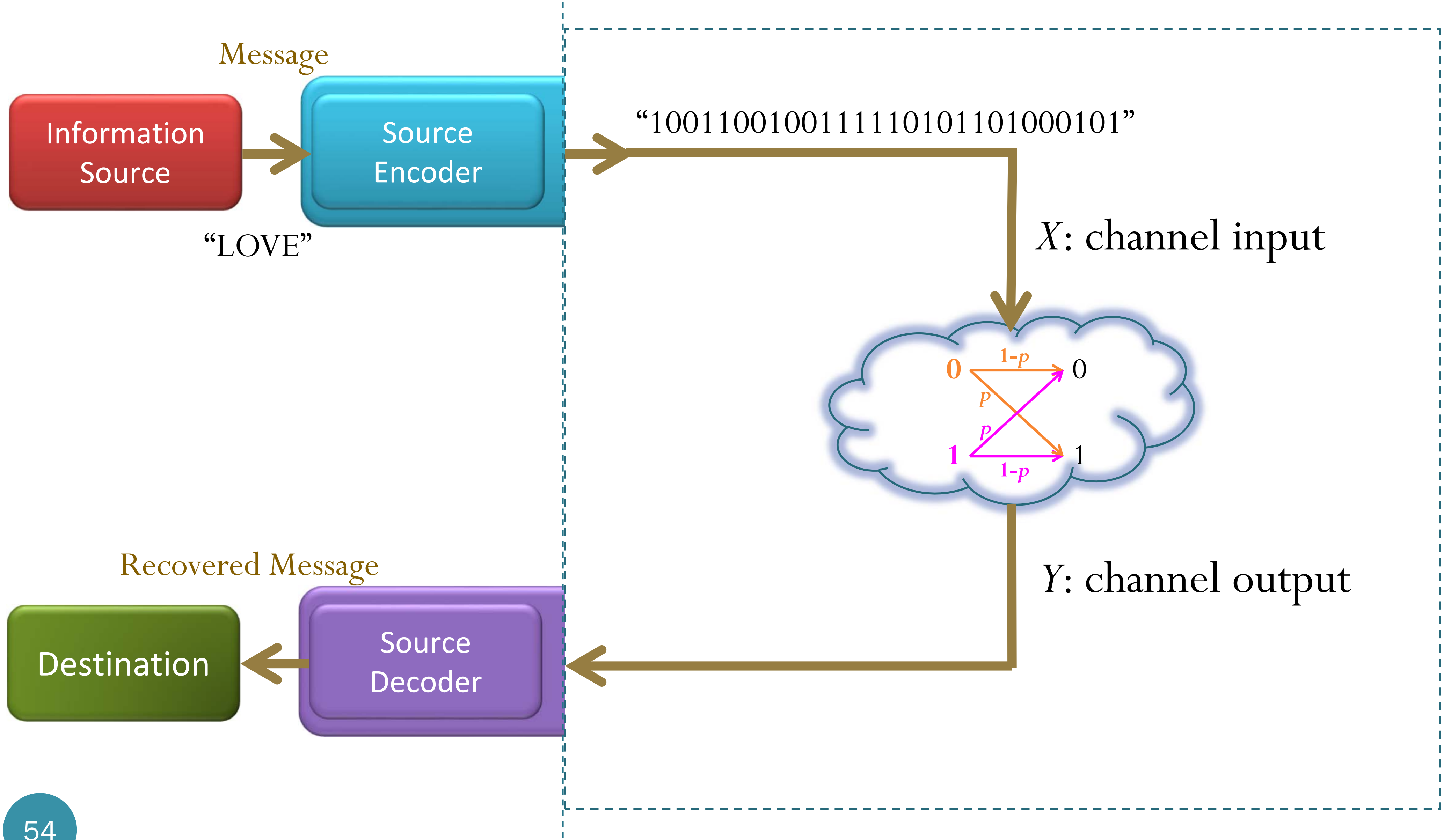


  - BSC with crossover probability $p = 0.01$
- New (and Better) Equivalent Channel:



  - Use repetition code with $n = 5$ at the transmitter
  - Use majority vote at the receiver
  - New BSC with $\tilde{p} = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5(1-p)^0 \approx 10^{-5}$

Example: ASCII Encoder and BSC



The ASCII Coded Character Set

Bit Number					6	0	0	0	0	1	1	1	1
					5	0	0	1	1	0	0	1	1
					4	0	1	0	1	0	1	0	1
				Hex	1st	0	1	2	3	4	5	6	7
3	2	1	0	2nd		0	16	32	48	64	80	96	112
0	0	0	0	0		NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1		SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2		STX	DC2	"	2	B	R	b	r
0	0	1	1	3		ETX	DC3	#	3	C	S	c	s
0	1	0	0	4		EOT	DC4	$	4	D	T	d	t
0	1	0	1	5		ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6		ACK	SYN	&	6	F	V	f	v
0	1	1	1	7		BEL	ETB	'	7	G	W	g	w
1	0	0	0	8		BS	CAN	(	8	H	X	h	x
1	0	0	1	9		HT	EM	)	9	I	Y	i	y
1	0	1	0	A		LF	SUB	*	:	J	Z	j	z
1	0	1	1	B		VT	ESC	+	;	K	[	k	{
1	1	0	0	C		FF	FS	,	<	L	\	l	|
1	1	0	1	D		CR	GS	-	=	M	]	m	}
1	1	1	0	E		SO	RS	.	>	N	^	n	~
1	1	1	1	F		SI	US	/	?	O	—	o	DEL

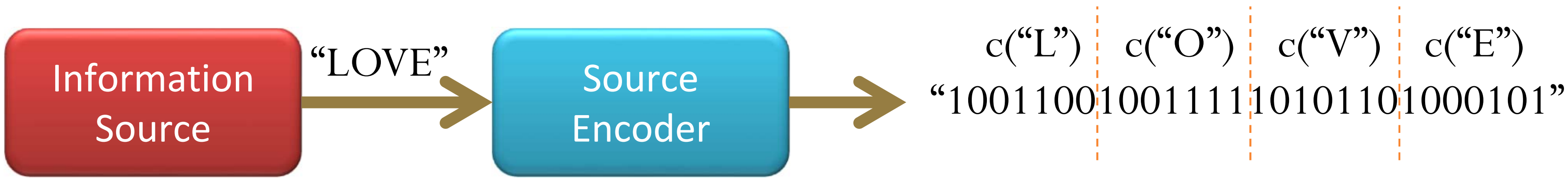
[The ARRL Handbook for Radio Communications 2013]

Example: ASCII Encoder

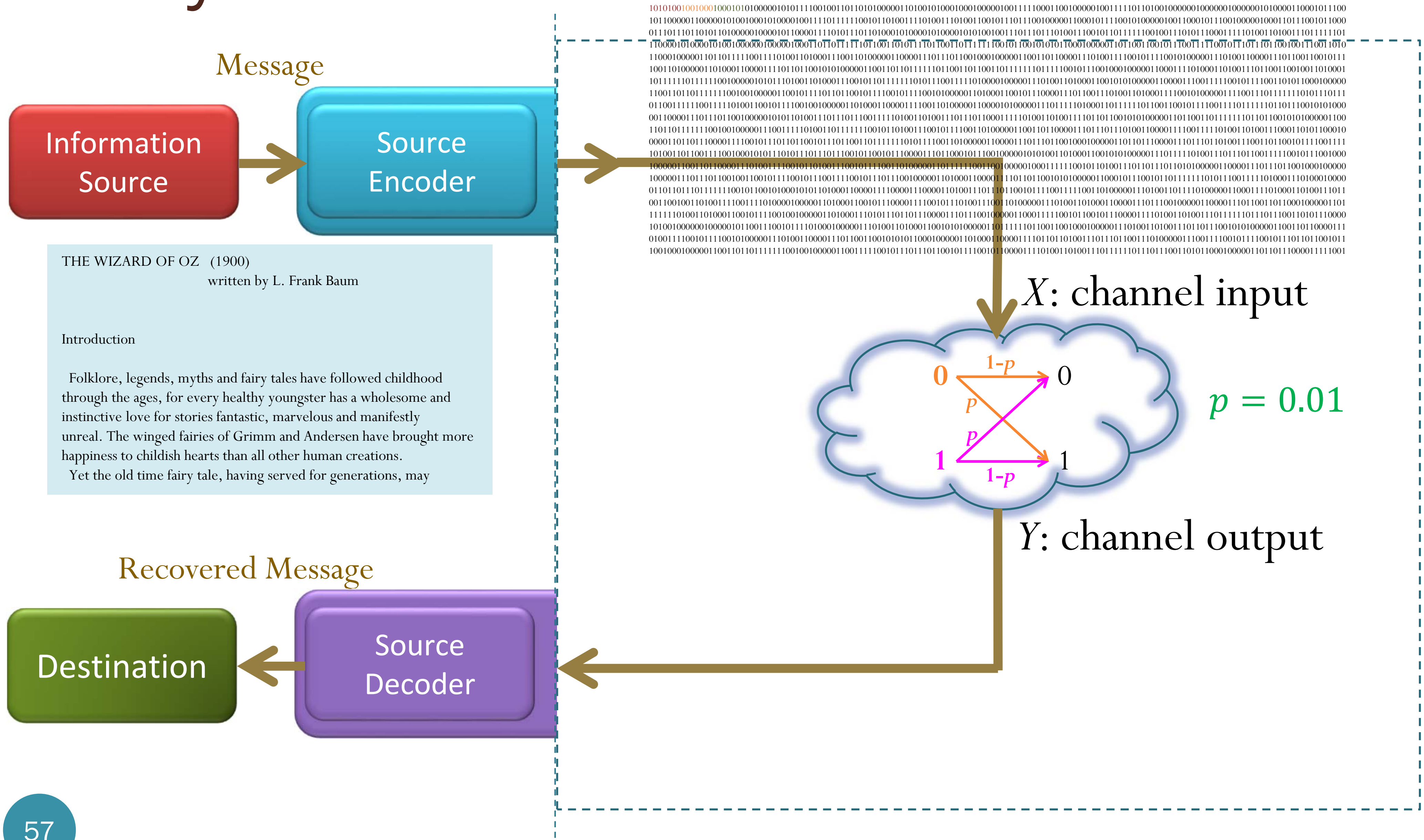
Character	Codeword
⋮	
E	1000101
⋮	
L	1001100
⋮	
O	1001111
⋮	
V	1010110
⋮	

MATLAB:

```
>> M = 'LOVE';
>> X = dec2bin(M,7);
>> X = reshape(X',1,numel(X))
X =
1001100100111110101101000101
```



System considered



[ErrorProbabilityoverBSC.m]

Results

THE WIZARD OF OZ (1900)
written by L. Frank Baum

Introduction

Folklore, legends, myths and fairy tales have followed childhood through the ages, for every healthy youngster has a wholesome and instinctive love for stories fantastic, marvelous and manifestly unreal. The winged fairies of Grimm and Andersen have brought more happiness to childish hearts than all other human creations.
Yet the old time fairy tale, having served for generations, may

THE WIZARD _F OZ (19009 written by L. Frank0Baum

Introduction

0Folklore. legendS myths and faiby talgs have fmllowed childhood through the ages, for$every nealthy youngster has a wholesome and ilspynctire love for storieq fa.tastic, marvelou3 end manifestly unreal. The winged fairies of Grimm and*Andersen havE brought more happiness to chihdish hearts than all odhur human creations/
Yet the0old"timm fai2y tale, having qerved for generationq, may

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Yet the0old"timm fai2y tale, having qerved for generationq, may

- The whole book which is saved in the file "OZ.txt" has 207760 characters (symbols).
- The ASCII encoded string has $207760 \times 7 = 1454320$ bits.
- The channel corrupts 14545 bits.
- This corresponds to 14108 erroneous characters.

Results

```
>> ErrorProbabilityoverBSC
biterror =
      14545
BER =
   0.010001237691842
theoretical_BER =
   0.010000000000000
characterErrror =
      14108
CER =
   0.067905275317674
theoretical_CER =
   0.067934652093010
```

- The file "OZ.txt" has 207760 characters (symbols).
- The ASCII encoded string has $207760 \times 7 = 1454320$ bits.
- The channel corrupts 14545 bits.
- This corresponds to 14108 erroneous characters.

$$\frac{14545}{1454320} \approx 0.01$$

$$\frac{14108}{207760} \approx 0.0679$$

Results

BSC's crossover probability

$$p = 0.01$$

$$\frac{14545}{1454320} \approx 0.01$$

$$\frac{14108}{207760} \approx 0.0679$$

$$\text{CER} = 1 - (1-p)^7$$

- The file "OZ.txt" has 207760 characters (symbols).
- The ASCII encoded string has $207760 \times 7 = 1454320$ bits.
- The channel corrupts 14545 bits.
- This corresponds to 14108 erroneous characters (symbols).

A character (symbol) is successfully recovered if and only if none of its bits are corrupted.

Crossover probability and readability

When the first novel of the series, Harry Potter and the Philosopher's Stone (published in some countries as Harry Potter and the Sorcerer's Stone), opens, it is apparent that some significant event has taken place in the wizarding world--an event so very remarkable, even the Muggles notice signs of it. The full background to this event and to the person of Harry Potter is only revealed gradually through the series. After the introductory chapter, the book leaps forward to a time shortly before Harry Potter's eleventh birthday, and it is at this point that his magical background begins to be revealed.

Original

When the first novel of the series, Harry Pottez and the Philosopher's Stone (p5blished in some countries as Harry Potter cnd the Sorcerep's Stone), opens, it i3 apparent that soMe cignifacant event!haS taken0place in the wi~arding 7orld--ao event so`very!bemark!blu, even the Mufgles nodice signs"of it. The fuld background to this event and to the person of Harry P/tTer is only revealed gradually through th series. After the introfuctory chapter, the boo+ leaps forward to a time shortly before Harpy Potteb7s eleventh`birthday, and)t is at this poi~t that his -agikal bac{ground begins to be revealed.

$p = 0.01$ CER ≈ 0.07

Crossover probability and readability

Human may be able to correct some (or even all) of these errors.

When the first novel of the series, Harry Pottez and the Philosopher's Stone (p5blished in some countries as Harry Potter cnd the Sorcerep's Stone), opens, it i3 apparent that soMe cignifacant event!haS taken0place in the wi~arding 7orld--ao event so`very!bemark!blu, even the Mufgles nodice signs"of it. The fuld background to this event and to the person of Harry P/tTer is only revealed gradually through th series. After the introfuctory chapter, the boo+ leaps forward to a time shortly before Harpy Potteb7s eleventh`birthday, and)t is at this poi~t that his -agikal bac{ground begins to be revealed.

$p = 0.01$ CER ≈ 0.07

Crossover probability and readability

w(en th% birst .ovo,`of the serieq, Hcrry Potter(and the Phidosop`er's Suone (xub|ishe$(in some!Countries as @arry Potter and t`e Snr#erer's S|ong)- opens, it is apparent thatsoee smgfificant erent ha3 taieN place in the wizardino world-,an event!so very remarkable, even thE Eugglds notiae qigns of it. Tledfull back'tound to this event ane to the perron of Harry Popter is onl{ reveqned gsadeally thro}gH th%$serias. After the int2oducpory chcptur, the0jook deaps forward to"a!tmme shmrtly befosE Harry"Potter's eleventh jirthdiy cnd ht is a| thi3 po{nt tHat @is mAgiial background begijs to rm rerealed.

$p = 0.02$ CER ≈ 0.13

Whethe first nOwgl nf the susi-Q-@Harr} PoutEr(and |he PhilosoxhEr's Ctonepuclyshed in som% coultries as Harrx @ tter and the S_rcermr7s Spone), opdns, id is apparent that {omg`signifikant evmnt ias taKen!Placa in tHe 7ijardIng world--an event so Very remaroqble, eve.!thE MugglaC fotice signc of"it. Uhe full backf2ound`to thas even| ant`0o the pEssoj of @arry Qotteb iw only revealed gradu!lly vhvoug` the rerier. Afte2 the IndRoductori chaptar,t`e book leats ForwaRf tc a 4imE shostl= before!Hcssy potter's u|Eveoth$firthdA{, and iT is ad this pomNt uhav `ir magica, back'bound cegins to bE 2evealed.

$p = 0.03$ CER ≈ 0.19

Crossover probability and readability

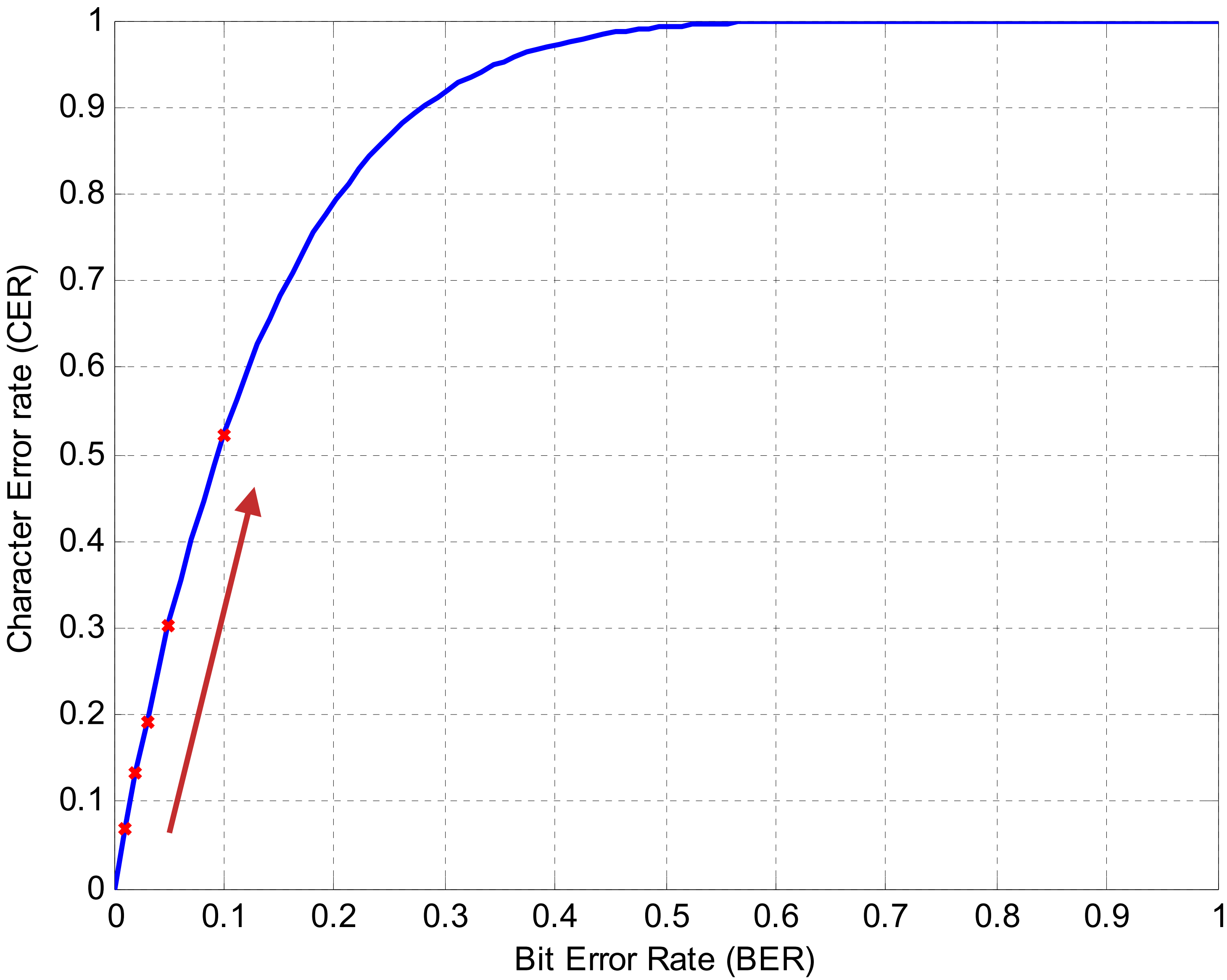
When phe fir   v okval ov"th% serie3, @`rry0Pntter efdxtxe Thil    soph%rs Stone0(p}blisjed!in {ooe c    un|pye{ agav0y Potter aj`(the sorcerer"s S4o|e)< opdns- mt"Is!apParEnt 4hat somu siwnidiga.v evant iAs take."plhge in(uhe w)zard)ng wo{|d--An event so very Rumar{ablel eteN0Dhe %ugcles$n   t)ce signs of$At. Tje!&ul|!backep/und Dk thkw`event ajt(to vhd per{On of8Ikxry P_Pter is oN,y rereAeud gredualli 4hroufh5ie qeriesn Af|ir the )~trofUctkry!ciapter,$tle r%ok lE`ps for   erd8to!a d)hg 3Hostly redobd HArry(Potter/r elaventI(birpl%ay,))nd(iD i3 1t tlis hohlt vhat iis$iagical bac+gropnd bedans to bg rEve!ied/

$p = 0.05$ CER ≈ 0.30

Whef th% &i2sv nkvdl"On(txE"serm-s< HaRtY Qo|p%R$anlthe Phi$)qop8gb'r YtoNe (puclirhed in23/ee c   uNpr9es aZ Harby!PovDdZ qnd0THA!Uorojev's Qpof'),   pegsL iT is0aqazenP Tiet`{nle sau*!fICQ~t eve.t`xA# raken pOqb%%)D }Hm`wizprdYjv"wOrnd--a~%W%Jv s' tury 2maskABdd$`eden(tl| LuxGxec`nOtike c)gzq of kt Tiu!f5mm"cackG@    Ud(to"vhhQ a~aNd alt tn0vid veRckn of HaRvq$Xntter#isxohk{ regea,ed@&saduadLy u(2otGh"tau griEs."AfTex0T`g mntr   DUCt   ry kh `ter,$thd(fomN0j`apv ngrwarTt-0c t,me"1xortly bEemsL |ar2q Pnfter'3 aMen-n5i@Fipth$`q, aoh It i3d1t piac0pmhnP d*if Zas mafibin"je#k7poUndpb%dins tk`be qe6e!lgd.

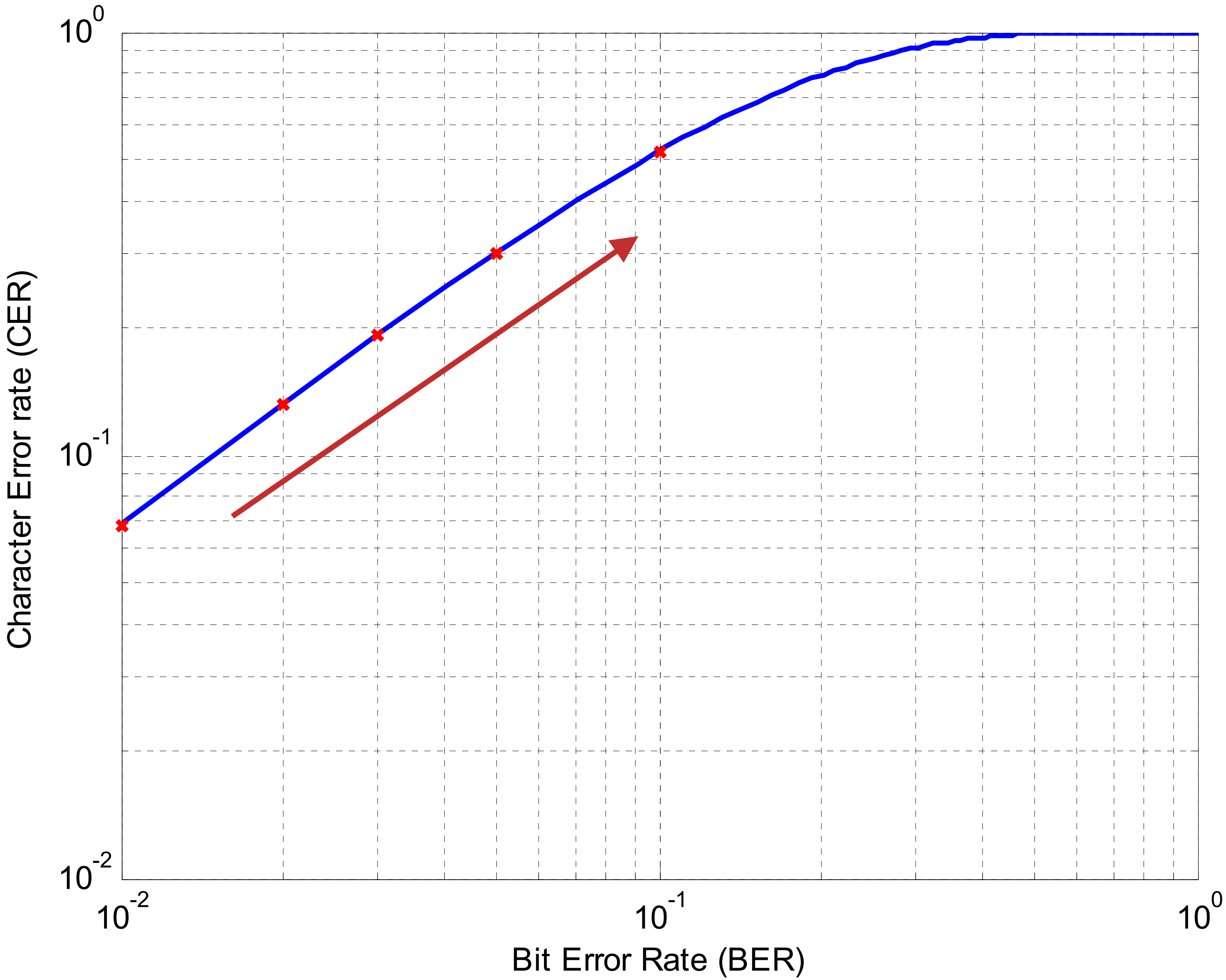
$p = 0.10$ CER ≈ 0.52

BER vs. CER



[BERCER.m]

BER vs. CER



[BERCER.m]